The Effect of a Weak Toroidal Field on the $n=2$ Rotational Instability in an FRC

Background: \( n=2 \) Rotational Instability

- Rotation drives a gravitation-like instability in FRCs
- This mode is analogous to the gravity driven Raleigh-Taylor instability: the centrifugal effect gives rise to gravity
- Historically, the rotational mode appears almost without exception in FRCs
- Although the feared tilt has drawn more attention, the rotational mode is the only consistently observed instability in FRCs
Stabilization of the rotational mode

- Multipole stabilization of rotational mode

\[ \frac{(B_\theta)_{qu}}{\sqrt{\mu_0 m_i n_m}} > \gamma_0 r_s \]  
(Ishimura 1984)

- Magnetic shear stabilization of gravitational mode

magnetic compressibility + field-line tension  
(e.g. Guo 2005)

\[ \frac{B_\theta}{\sqrt{\mu_0 m_i n_0}} > \frac{1}{\pi} \gamma_g r_s \]

Growth rate of grav'l mode, \( \gamma_g = (g/L_n)^{1/2} \)

- Suggested rotational stability parameter:

\[ Q \equiv \frac{B_{\theta_0}}{\sqrt{\mu_0 m_i n_m R \Omega}} \]

Toroidal field at magn. axis
Rotation frequency
Radius of magn. axis
Dispersion relation

(Frieman-Rotenberg, 1958)

\[-\omega^2 K + \omega P + W_g + W_B = 0\]

- "Reactive" term missing from strictly "gravitational" analyses, gives rise to…
  (a) real frequency component, \(\omega_R\)
  (b) reactive stabilization effect

- Gravitational term drives the instability

- Magnetic field term (toroidal field): potentially stabilizing
Elements of analytic model

- Highly-elongated FRC, $\partial/\partial z = 0$

- Toroidal field structure: $B_\theta = B_0 b_\theta(x)$
  
  $$b_\theta = \begin{cases} 
  x^{1/2} (2 - x); & 0 \leq x \leq 2 \\
  0; & 2 < x
  \end{cases}$$

- Hill’s-vortex-like density structure

  $$n = n_m \begin{cases} 
  1 - (1 - N_s)(1 - x)^2; & 0 \leq x \leq 2 \\
  N_f; & 2 < x
  \end{cases}$$

- Displacement trial function

  $$\xi_r = x^{1/2} f(x; \ldots) \quad \xi_\theta = (i/m) d(r\xi_r)/dr$$

- Toroidal field at magn. axis

- Separatrix density

- Multi-parameter trial function

- Assures $\nabla \cdot \xi = 0$
Analytic stability for trial function

- Assume simple trial function with linear dependence on radius $\xi_r = r/R$

- $K' = 4$, $P' = -8$, $W'_G = -8$, $W'_B = 16/3 \rightarrow Q_\ast = (3/4)^{1/2}$

- Marginal stability condition

$$B_{\theta \text{max}} / \sqrt{\mu_0 m_i n_m} > 0.61 r_s \Omega$$
Quasi-analytic stability criterion

- Flexible trial function: \( \xi_r = x^{1/2} f(x) \)

\[
f(x) = \left[ \alpha + (1-\alpha) \frac{x}{x_1} \right] \left[ 1 - e^{\sigma(x-x_1)} \right]^2
\]

- Three parameters:
  \( x_1 \) - radial “edge” of mode
  \( \alpha \) - flattening near \( r = 0 \)
  \( \sigma \) - sharpness of fall-off near edge
Quasi-analytic stability criterion

Basic stability criterion

\[ B_{\theta_{\text{max}}} / \sqrt{\mu_0 m_i n_m} > \frac{0.77}{(4/3\sqrt{3})Q^* r_s \Omega} \]

- \( Q^* \) determined by computation of \( K, P, W_G, W_B \)

- This criterion resembles stability criterion for
  - Quadrupole stabilization of rotational mode
  - Magnetic shear stabilization of gravitational mode
Quasi-analytic stability criterion (cont.)

- Dependence on parameter $\alpha$

- *Typical* $\gamma\Omega \sim 1$
  $$\omega_R/\Omega \sim -1$$
  $$Q^* \sim 5/6$$

- Stability condition
  $$B_{\theta_{\text{max}}} / \sqrt{\mu_0 m_i n_m} > 0.64 r_s \Omega$$

Compare this proportionality factor to simple analytical result: 0.61
Method Using NIMROD Code

- Start with non-rotating equilibrium, add a rigid rotation and toroidal magnetic field to the n=0 component.
  - Let NIMROD seek a new equilibrium
- Running MHD, with and without the Hall term
  - With and without toroidal fields
  - Rotation rate: \( \Omega_i \sim \Omega_{di} = 4KT_i / (R^2B) \)
- Initialize with n=2 perturbation in \( u_r \) the FRC
- Add a toroidal field \( rB_\theta \sim \psi/\psi_o \) inside the FRC and zero outside.
Calculation Parameters

- \( r_w = 0.4 \text{ m}, x_s = 0.65, r_s = 0.26 \text{ m}, z_s = 1.5 \text{ m} \)
- \( n_o = 1.24 \times 10^{20} \text{ m}^{-3}, T_e = T_i = 100 \text{ eV}, B_e = 0.1 \text{ T} \)
- Elongation = 5.8, \( S^* = 9.0, E/S^* = 0.64 \)
- Isotropic viscosity \( \nu = 100 \text{ m}^2/\text{s} \),
  - Reynolds num \( \sim 400 \)
- \( \eta/\mu_o = 5 \text{ m}^2/\text{s} \),
  - Lundquist num \( \sim 8000 \)
Density Evolution for Base Case (no $B_\theta$)

- Rotation rate: $\Omega_{di} = 4KT_i / (R^2B) = 1.09 \times 10^5$ $\Omega_i = 10^5$
- Growth rate: $\gamma/\Omega_i = 0.74$
Imposed Toroidal Field Profile

\[ rB_\theta \sim \psi/\psi_o \] inside the FRC, and zero outside.
NIMROD Comparison with Analysis

- Run with **no axial dependence (r-θ) only**
  - Run with short section at midplane and periodic boundary conditions.
  - Stable for $B_\theta/B_e > 0.2$.

![Growth of n=2 mode as a function of time](image-url)
Evolution of Mode Structure

- The radial velocity profile peak moves out radially as the toroidal field strength is increased.
  - The peak shifts to open field lines, where $B_\theta=0$
Evolution of Mode Structure

- The mode moves out radially as toroidal field increases.
- Boundary condition sets $V_r$ and $V_\theta$ to zero at the wall.

\[
\mathbf{B}_{\theta} = 0.02
\]

Velocity ~ divergence free $\nabla \cdot \mathbf{v} = 0 \Rightarrow \nu_\theta = \frac{i}{n} \frac{\partial}{\partial r} \left( r \nu_r \right).

Peak velocity near separatrix
Mode activity pushed outside separatrix
Comparison with analysis

- Reference case: $B_e = 0.1$ T, $r_s = 0.26$ m,
  $\Omega = 10^5$ rad/s, $n_m = 1.24 \times 10^{20}$ m$^{-3}$

- Nominal stability condition for conventional rotational mode (NIMROD):
  $B_{\theta,\text{max}} > 0.013$ T

- Marginal stability condition from analysis
  $B_{\theta,\text{max}} > 0.012$ T (about 10% too optimistic)

- Analysis based on trial-function approach, yet gives a quite good approximation of stability
Growth of n=2 mode with and without toroidal field (Elongation = 6.0)

- With toroidal field, growth is delayed as initial perturbation is not aligned with fastest mode.
Mode Structure for Base Case (no $B_\theta$)

Velocity ($n=2$) at the axial midplane at 50 µsec.
Boundary condition sets $V_r$ and $V_\theta$ to zero at the wall.
Mode Structure for Base Case (no $B_\theta$)

- Velocity vectors (n-2) in r-z plane at 50 µsec
  - $\theta$ chosen for plane of maximum velocity
- Velocity is dominantly in the r-$\theta$ direction
  - There is a significant axial velocity component on the open field-lines near the end of the FRC
Mode Structure for Case with $B_\theta = 0.03$

Velocity (n=2) at the axial midplane at 100 µsec.

Compared to the calculation with no toroidal field:
- Mode structure has moved out radially (where $B_\theta=0$)
- A significant axial velocity is induced
Mode Structure for Case with $B_\theta = 0.03$

- Velocity vectors (n-2) in r-z plane at 100 µsec
  - $\theta$ chosen for plane of maximum velocity
- Velocity is dominantly on the open field lines
- Axial velocity has components both even and odd in z.
Velocity Vectors: $n=2$ Component in Plane of Maximum Perturbation

- $B_\theta = 0.0$, No Hall term, $t=50$ µsec
- $B_\theta = 0.03$T, No Hall term, $t=98$ µsec
- $B_\theta = 0.03$T, Hall term on, $t=98$ µsec
Summary of some key parameters

◆ Elongation = 6.0

$E_{\text{FRC}}/E_{\text{Tot}}$ shows that adding $B_\theta$ causes the mode energy to move to region outside FRC.

$E_z/E_r$ (FRC) shows that adding $B_\theta$ causes the mode to evolve to one with significantly more axial velocity.

<table>
<thead>
<tr>
<th>$B_\theta/B_e$</th>
<th>Hall</th>
<th>$\gamma/\Omega$</th>
<th>$E_{\text{FRC}}/E_{\text{Tot}}$</th>
<th>$E_z/E_r$ (FRC)</th>
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<tr>
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<td>0.74</td>
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<td>0.575</td>
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<td>No</td>
<td>0.51</td>
<td>0.23</td>
<td>1.5</td>
</tr>
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<td>No</td>
<td>0.46</td>
<td>0.2</td>
<td>4</td>
</tr>
<tr>
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<td>0.15</td>
<td>14</td>
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<td>0.68</td>
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<td>0.25</td>
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</table>
Growth of n=2 mode for shorter FRC (Elongation = 3.8)

- In the shorter elongation case, axial modes grow faster, but the Hall term more effectively stabilizes them.
Velocity Vectors: $n=2$ Component in Plane of Maximum Perturbation

- $B_\theta=0.0$, No Hall term, $t=50 \, \mu\text{sec}$
- $B_\theta=0.03\,\text{T}$, No Hall term, $t=97 \, \mu\text{sec}$
- $B_\theta=0.03\,\text{T}$, Hall term on, $t=97 \, \mu\text{sec}$
Summary of some key parameters

◆ Elongation = 3.8

<table>
<thead>
<tr>
<th>$B_\theta/B_e$</th>
<th>Hall</th>
<th>$\gamma/\Omega$</th>
<th>$E_{FRC}/E_{Tot}$</th>
<th>$E_z/E_r$ (FRC)</th>
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<tbody>
<tr>
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<td>0.84</td>
<td>0.65</td>
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<td>0.1</td>
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<td>&gt; 50</td>
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<td>&gt; 40</td>
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$E_{FRC}/E_{Tot}$ shows that adding $B_\theta$ causes the mode energy to move to region outside FRC.

$E_z/E_r$ (FRC) shows that adding $B_\theta$ causes the mode to evolve to one with significantly more axial velocity.

Effect more dramatic for shorter FRC.
Summary: n=2 Rotational Instability

◆ Modest toroidal field can stabilize the conventional n=2 rotational instability: $B_{\theta_{\text{max}}} / \sqrt{\mu_0 m_i n_m} > 0.61 r_s \Omega$

◆ NIMROD calculations are complicated by:
  – n=2 axial modes that are not observed experimentally and presumably stabilized by kinetic effects.
  – Open field line plasma is also n=2 unstable (but could be stabilized by expected shear flow).

◆ Hall term:
  – Almost no effect on conventional n=2 mode.
  – Dramatic stabilizing effect on n=2 axial modes.