Recent results from the general moment method

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Overview

- General moment equations
  - Linearized collision operators Ji Held, PoP 2006 2008
  - Nonlinear collision terms Ji Held PoP 2009
- Analytic solution for a uniform plasma Ji Held, Phys. Rev. E 2010
- Parallel closures for general collisionality
  - Heat flow Ji Held Sovinec, PoP 2009
  - Viscosity Ji Held, J Fusion Energy 2009
- High collisionality closures for electron-ion plasmas
  - Electrons: require more moments
  - Ions: keep ion-electron operator and time derivative terms
- General collisionality closures in a strong magnetic field NIMROD meeting October 2009

General moment equations

- Landau (Fokker-Planck) kinetic equation

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\cdot\nabla f_a + \frac{q_a}{m_a}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_a}{\partial \mathbf{v}} = \sum_b C(f_a, f_b) \quad (+X+I+R)$$

- Moment expansion

$$f_a(t,\mathbf{x},\mathbf{v}) = f_a^{\mathrm{M}} \sum_{lk} \mathsf{m}_a^{lk}(t,\mathbf{x}) \cdot \hat{\mathsf{p}}_a^{lk} \text{ with } f_a^{\mathrm{M}} = \frac{n_a}{\pi^{3/2} v_{Ta}^3}\exp(-c_a^2)$$

$$\mathsf{n}_a^{lk} \equiv n_a \mathsf{m}_a^{lk}(t,\mathbf{x}) = \int d\mathbf{v}\, \hat{\mathsf{p}}_a^{lk} f_a(t,\mathbf{x},\mathbf{v})$$

where $\hat{\mathsf{p}}^{lk}$'s are orthonormal polynomials of $\mathbf{c}_a = (\mathbf{v}-\mathbf{V}_a)/v_{Ta}$

- General moment equations: $\int d\mathbf{v}\,\hat{\mathsf{p}}^{jp}$(kinetic eq.)$\Rightarrow$

$$d_a \mathsf{n}_a^{jp} + \Omega_a \mathbf{b}\breve{\times}\mathsf{n}_a^{jp} + \{\hat{\Xi}^j(d_a \ln T) + \hat{U}_{\mathrm{c}}^j \nabla\cdot\mathbf{V} + \hat{U}_{\mathrm{l}}^j(\underline{\nabla}\mathbf{V})\cdot + \hat{U}_{\mathrm{r}}^j(\nabla\underline{\mathbf{V}})\cdot\}_{pk}\mathsf{n}_a^{jk}$$
$$+ \{v_T\hat{\Psi}^{j\pm}\nabla + v_T\hat{\Phi}^{j\pm}\nabla\ln T + v_T^{-1}\hat{\Theta}^{j\pm}\mathbf{a}_a\}_{pk}(\cdot)\mathsf{n}_a^{j\pm1,k}$$
$$+ \hat{U}_{pk}^{j\pm}\nabla\mathbf{V}(:)\mathsf{n}_a^{j\pm2,k} = (\hat{C}_{aa}^{jpk} + \hat{A}_{ab}^{jpk})\mathsf{n}_a^{jk} + \hat{B}_{ab}^{jpk}\mathsf{n}_b^{jk} + C_{ab}^{(2)jp}$$

where $d_a \equiv \partial_t + \mathbf{V}_a\cdot\nabla$ and $\mathbf{a}_a \equiv \dfrac{q_a}{m_a}(\mathbf{E}+\mathbf{V}_a\times\mathbf{B}) - d_a\mathbf{V}_a$

- ▶ Farewell to the velocity variable $\mathbf{v}$

Several low order moments

$\mathsf{p}^{lk} = \mathsf{P}^l(\mathbf{c})L_k^l(c^2)$	$L_k^l = L_k^{(l+\frac{1}{2})}$	n^{lk}	fluid moment equation	indep.
$\mathsf{P}^0 = 1$	$L_0^0 = 1$	n	density	1
	$L_1^0 = \frac{3}{2} - c^2$	0	temperature	1
$\mathsf{P}^1 = \mathbf{c}$	$L_0^1 = 1$	0	flow velocity	3
	$L_1^1 = \frac{5}{2} - c^2$	n^{11}	heat flow $(\mathbf{h})$	3
	$L_2^1 = \frac{35}{8} - \frac{7}{2}c^2 + \frac{1}{2}c^4$	n^{12}	heat w. heat flow	3
$\mathsf{P}^2 = \mathbf{cc} - \frac{c^2}{3}\mathsf{I}$	$L_0^2 = 1$	n^{20}	viscosity $(\boldsymbol{\pi})$	5
	$L_1^2 = \frac{7}{2} - c^2$	n^{21}	heat viscosity	5

Moment equations for closures

- Maxwellian moment $(n_a, \mathbf{V}_a, T_a)$ equations

$$(0,0): \quad d_a n_a + n_a \nabla \cdot \mathbf{V}_a = 0$$

$$(0,1): \quad \tfrac{3}{2} n_a d_a T_a + n_a T_a \nabla \cdot \mathbf{V}_a + \nabla \cdot \mathbf{h}_a + \nabla \mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$$

$$(1,0): \quad \underbrace{m_a n_a d_a \mathbf{V}_a - n_a q_a (\mathbf{E} + \mathbf{V}_a \times \mathbf{B})}_{-m_a n_a \mathbf{a}_a} + \nabla p_a + \nabla \cdot \boldsymbol{\pi}_a = \mathbf{R}_a$$

- Non-Maxwellian moment equations $(j,p) \neq (0,0), (0,1), (1,0)$

$$\hat{D}_a \mathsf{n}_a + \Omega_a \mathbf{b} \breve{\times} \mathsf{n}_a = (\hat{C}_{aa} + \hat{A}_{ab}) \mathsf{n}_a + \mathsf{G}_a + \hat{B}_{ab} \mathsf{n}_b + \hat{C}_a^{(2)}$$

where $\mathsf{n}_a = (\mathsf{n}_a^{02}, \mathsf{n}_a^{03}, \cdots, \mathsf{n}_a^{11}, \mathsf{n}_a^{12}, \cdots, \mathsf{n}_a^{20}, \mathsf{n}_a^{21}, \cdots, \cdots)^{\mathrm{T}}$,

$$\mathsf{G}_a^1 = \begin{pmatrix} \frac{\sqrt{5}}{2} \frac{n_a v_{Ta}}{T_a} \nabla T_a + \delta_{ae} \sqrt{2} a_{\mathrm{ei}}^{110} \frac{n_{\mathrm{e}}}{\tau_{\mathrm{ei}}} \frac{\mathbf{V}_{\mathrm{ei}}}{v_{T\mathrm{e}}} \\ \delta_{ae} a_{\mathrm{ei}}^{120} \frac{n_{\mathrm{e}}}{\tau_{\mathrm{ei}}} \frac{\mathbf{V}_{\mathrm{ei}}}{v_{T\mathrm{e}}} \\ \delta_{ae} a_{\mathrm{ei}}^{130} \frac{n_{\mathrm{e}}}{\tau_{\mathrm{ei}}} \frac{\mathbf{V}_{\mathrm{ei}}}{v_{T\mathrm{e}}} \\ \vdots \end{pmatrix}, \quad \mathsf{G}_a^2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} n_a \mathsf{W}_a \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

$a_{\mathrm{ei}}^{1p0} = -\sqrt{\frac{3(p+\frac{1}{2})!}{(2p+3)p!(\frac{1}{2})!}}$, and $\mathsf{W} = \nabla \mathbf{V} + (\nabla \mathbf{V})^{\mathrm{T}} - \frac{2}{3} \mathsf{I} \nabla \cdot \mathbf{V}$

General moment equations

$\Rightarrow$ 21 moment (5+N{$\mathsf{n}^{11},\mathsf{n}^{12},\mathsf{n}^{20},\mathsf{n}^{21}$}) equations (here $\dot{\nabla}\equiv v_T\nabla$)

$$d_t\mathsf{n}^{11}-\sqrt{\tfrac{2}{5}}\dot{\nabla}\cdot\mathsf{n}^{20}+\sqrt{\tfrac{7}{5}}\dot{\nabla}\cdot\mathsf{n}^{21}-\tfrac{\sqrt{5}}{2}n\dot{\nabla}\ln T+\Omega\mathbf{b}\times\mathsf{n}^{11}=\hat{C}^1_{10}\hat{\mathbf{V}}_{\mathrm{ei}}+\hat{C}^1_{11}\mathsf{n}^{11}+\hat{C}^1_{12}\mathsf{n}^{12}$$

$$(\text{l.h.s.}):-\sqrt{\tfrac{2}{3}}\dot{\nabla}\mathsf{n}^{02}+\sqrt{\tfrac{2}{5}}\tfrac{2}{v_T}\mathbf{a}\cdot\mathsf{n}^{20}+\tfrac{3}{2}d_t\ln T\mathsf{n}^{11}+\dot{\nabla}\ln T\cdot(-\tfrac{9}{\sqrt{10}}\mathsf{n}^{20}+\sqrt{\tfrac{28}{5}}\mathsf{n}^{21})$$
$$+\tfrac{7}{5}\nabla\cdot\mathbf{V}\mathsf{n}^{11}+\tfrac{7}{5}\underline{\nabla}\mathbf{V}\cdot\mathsf{n}^{11}+\tfrac{2}{5}\nabla\underline{\mathbf{V}}\cdot\mathsf{n}^{11}-\sqrt{\tfrac{8}{3}}\dot{\nabla}\ln T\mathsf{n}^{02}-\sqrt{\tfrac{12}{5}}\nabla\mathbf{V}:\mathsf{n}^{30}$$

$$d_t\mathsf{n}^{12}-\tfrac{2}{\sqrt{5}}\dot{\nabla}\cdot\mathsf{n}^{21}+\sqrt{\tfrac{7}{6}}\dot{\nabla}\mathsf{n}^{02}-\dot{\nabla}\mathsf{n}^{03}+\tfrac{3}{\sqrt{5}}\dot{\nabla}\cdot\mathsf{n}^{22}+\Omega\mathbf{b}\times\mathsf{n}^{12}=\hat{C}^1_{20}\hat{\mathbf{V}}_{\mathrm{ei}}+\hat{C}^1_{21}\mathsf{n}^{11}+\hat{C}^1_{22}\mathsf{n}^{12}$$

$$(\text{l.h.s.}):+\tfrac{4}{\sqrt{5}v_T}\mathbf{a}\cdot\mathsf{n}^{21}+d_t\ln T(-\sqrt{7}\mathsf{n}^{11}+\tfrac{5}{2}\mathsf{n}^{12})+\dot{\nabla}\ln T\cdot(\sqrt{\tfrac{14}{5}}\mathsf{n}^{20}-\tfrac{13}{\sqrt{5}}\mathsf{n}^{21}+\tfrac{9}{\sqrt{5}}\mathsf{n}^{22})$$
$$-\tfrac{2\sqrt{7}}{5}\nabla\cdot\mathbf{V}\mathsf{n}^{11}-\tfrac{2\sqrt{7}}{5}\underline{\nabla}\mathbf{V}\cdot\mathsf{n}^{11}-\tfrac{2\sqrt{7}}{5}\nabla\underline{\mathbf{V}}\cdot\mathsf{n}^{11}+\tfrac{9}{5}\nabla\cdot\mathbf{V}\mathsf{n}^{12}+\tfrac{9}{5}\underline{\nabla}\mathbf{V}\cdot\mathsf{n}^{12}+\tfrac{4}{5}\nabla\underline{\mathbf{V}}\cdot\mathsf{n}^{12}$$
$$+\nabla\mathbf{V}:(\sqrt{\tfrac{48}{35}}\mathsf{n}^{30}-\sqrt{\tfrac{216}{35}}\mathsf{n}^{31})-\sqrt{\tfrac{14}{3}}\tfrac{1}{v_T}\mathbf{a}\mathsf{n}^{02}+\dot{\nabla}\ln T(\sqrt{\tfrac{175}{6}}\mathsf{n}^{02}-3\mathsf{n}^{03})$$

$$d_t\mathsf{n}^{20}-\sqrt{\tfrac{2}{5}}\dot{\nabla}\mathsf{n}^{11}+\sqrt{\tfrac{3}{2}}\dot{\nabla}\cdot\mathsf{n}^{30}+\sqrt{2}n\overline{\nabla\mathbf{V}}+\Omega\mathbf{b}\breve{\times}\mathsf{n}^{20}=\hat{C}^2_{00}\mathsf{n}^{20}+\hat{C}^2_{01}\mathsf{n}^{21}$$

$$(\text{l.h.s.}):+d_t\ln T\mathsf{n}^{20}-\tfrac{3}{\sqrt{10}}\dot{\nabla}\ln T\mathsf{n}^{11}+\nabla\cdot\mathbf{V}\mathsf{n}^{20}+2\underline{\nabla}\mathbf{V}\cdot\mathsf{n}^{20}+\sqrt{\tfrac{27}{8}}\nabla\ln T\cdot\mathsf{n}^{30}$$

$$d_t\mathsf{n}^{21}+\sqrt{\tfrac{7}{5}}\dot{\nabla}\mathsf{n}^{11}-\tfrac{2}{\sqrt{5}}\dot{\nabla}\mathsf{n}^{12}-\sqrt{\tfrac{3}{7}}\dot{\nabla}\cdot\mathsf{n}^{30}+\sqrt{\tfrac{27}{14}}\dot{\nabla}\cdot\mathsf{n}^{31}+\Omega\mathbf{b}\breve{\times}\mathsf{n}^{21}=\hat{C}^2_{10}\mathsf{n}^{20}+\hat{C}^2_{11}\mathsf{n}^{21}$$

$$(\text{l.h.s.}):-\sqrt{\tfrac{7}{5}}\tfrac{2}{v_T}\mathbf{a}\mathsf{n}^{11}+\dot{\nabla}\ln T(\tfrac{7}{2}\sqrt{\tfrac{7}{5}}\mathsf{n}^{11}-\sqrt{5}\mathsf{n}^{12})+d_t\ln T(2\mathsf{n}^{21}-\sqrt{\tfrac{7}{2}}\mathsf{n}^{20})$$
$$-\sqrt{\tfrac{2}{7}}\nabla\cdot\mathbf{V}\mathsf{n}^{20}-\sqrt{\tfrac{8}{7}}\underline{\nabla}\mathbf{V}\cdot\mathsf{n}^{20}-\sqrt{\tfrac{8}{7}}\nabla\underline{\mathbf{V}}\cdot\mathsf{n}^{20}+\tfrac{9}{7}\nabla\cdot\mathbf{V}\mathsf{n}^{21}+\tfrac{18}{7}\underline{\nabla}\mathbf{V}\cdot\mathsf{n}^{21}+\tfrac{4}{7}\nabla\underline{\mathbf{V}}\cdot\mathsf{n}^{21}$$
$$+\sqrt{\tfrac{3}{7}}\tfrac{2}{v_T}\mathbf{a}\cdot\mathsf{n}^{30}-\dot{\nabla}\ln T\cdot(\sqrt{\tfrac{108}{7}}\mathsf{n}^{30}+\tfrac{15}{2}\sqrt{\tfrac{3}{14}}\mathsf{n}^{31})-\sqrt{\tfrac{56}{15}}\nabla\mathbf{V}\mathsf{n}^{02}-\sqrt{\tfrac{24}{7}}\nabla\mathbf{V}:\mathsf{n}^{40}$$

Ordering for electron transport $\epsilon \sim \frac{v_T \tau_{ee}}{|\nabla|^{-1}} \ll 1,\ \mu \equiv \frac{m_e}{m_i},\ \hat{\mathbf{V}} \equiv \frac{\mathbf{V}_{ei}}{v_{Te}}$

- Transport ordering $\frac{\partial f}{\partial t} = O(\epsilon^2) \Rightarrow \partial_t \mathsf{n}_e^{jp},\ \partial_t \mathbf{V}_e(\cdot)\mathsf{n}_e^{j\pm 1,p},\ (\partial_t \ln T_e)\mathsf{n}_e^{jp}$

$$\frac{3}{2} n_e d_e \ln T_e + \cdots = 3\frac{m_e}{m_i}\frac{n_e}{\tau_{ei}}(\frac{T_i}{T_e} - 1) - \frac{1}{T_e}\mathbf{V}_{ei}\cdot\mathbf{R}_e$$

$$\underbrace{m_e n_e d_e \mathbf{V}_e + n_e e(\mathbf{E} + \mathbf{V}_e \times \mathbf{B})}_{-m_e n_e \mathbf{a}_e} + \cdots = -\frac{m_e n_e}{\tau_{ei}}\mathbf{V}_{ei} + \cdots$$

$$\underbrace{d_e \mathsf{n}_e^{jp}}_{\epsilon^2} + \Omega_e \mathbf{b}\breve{\times}\mathsf{n}_e^{jp} + \{\hat{\Xi}^j \underbrace{(d_e \ln T)}_{\mu,\hat{V}^2} + \underbrace{\hat{U}_c^j \nabla\cdot\mathbf{V} + \hat{U}_l^j(\underline{\nabla}\mathbf{V})\cdot + \hat{U}_r^j(\nabla\underline{\mathbf{V}})\cdot}_{\epsilon}\}_{pk}\mathsf{n}_e^{jk}$$

$$+\{\underbrace{v_T \hat{\Psi}^{j\pm}\nabla + v_T \hat{\Phi}^{j\pm}\nabla \ln T}_{\epsilon} + \hat{\Theta}^{j\pm}\underbrace{v_T^{-1}\mathbf{a}_e}_{\hat{V}}\}_{pk}(\cdot)\mathsf{n}_e^{j\pm 1,k}$$

$$+\hat{U}_{pk}^{j\pm}\underbrace{\nabla\mathbf{V}}_{\epsilon}(:)\mathsf{n}_e^{j\pm 2,k} = \frac{1}{\tau_{ee}}(c_{ee}^{jpk} + Z a_{ei}^{jpk})\mathsf{n}_e^{jk} + G_e^{jp} + \underbrace{\hat{B}_{ei}^{jpk}}_{\mu}\mathsf{n}_i^{jk} + \underbrace{C_e^{(2)jp}}_{\epsilon^2}$$

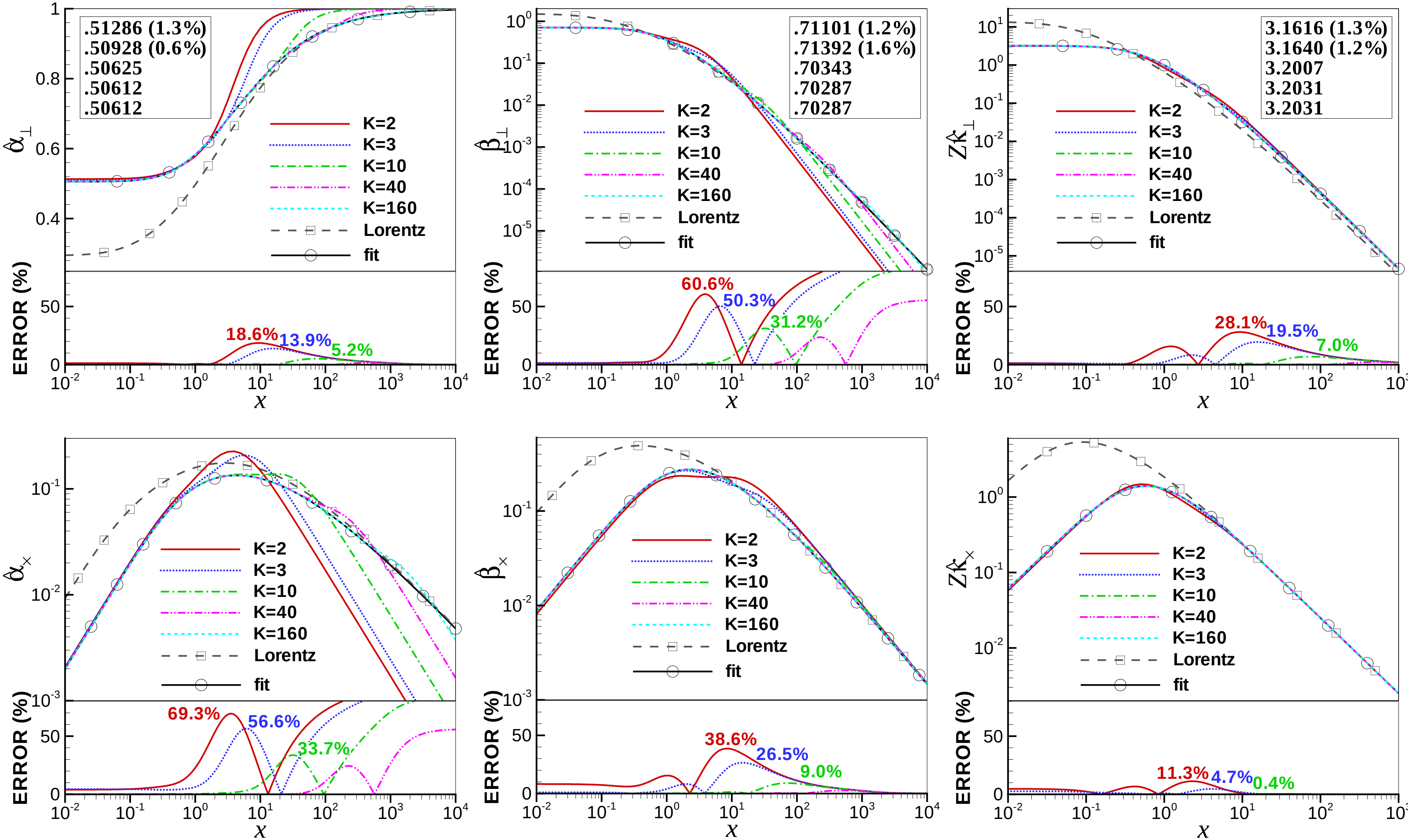
- Moment equations (2×2) for high collisionality $(r = \Omega_e \tau_{ee},\ \hat{\nabla} = v_T \tau_{ee}\nabla)$

$$\begin{pmatrix} -r\mathbf{b}\times\mathsf{n}^{11} \\ -r\mathbf{b}\times\mathsf{n}^{12} \end{pmatrix} = \begin{pmatrix} c_{11}^1 & c_{12}^1 \\ c_{21}^1 & c_{22}^1 \end{pmatrix}\begin{pmatrix} \mathsf{n}^{11} \\ \mathsf{n}^{12} \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{5}}{2}\hat{\nabla}\ln T + \sqrt{2} Z a_{ei}^{110}\hat{\mathbf{V}} \\ \sqrt{2} Z a_{ei}^{120}\hat{\mathbf{V}} \end{pmatrix}$$

Convergence study with increasing moments $(Z = 1)$

$$\mathbf{R}_\mathrm{e} = -\alpha_\parallel \mathbf{V}_{\mathrm{ei}\parallel} - \alpha_\perp \mathbf{V}_{\mathrm{ei}\perp} + \alpha_\times \mathbf{V}_{\mathrm{ei}\times} - \beta_\parallel \nabla_\parallel T_\mathrm{e} - \beta_\perp \nabla_\perp T_\mathrm{e} - \beta_\times \nabla_\times T_\mathrm{e}$$

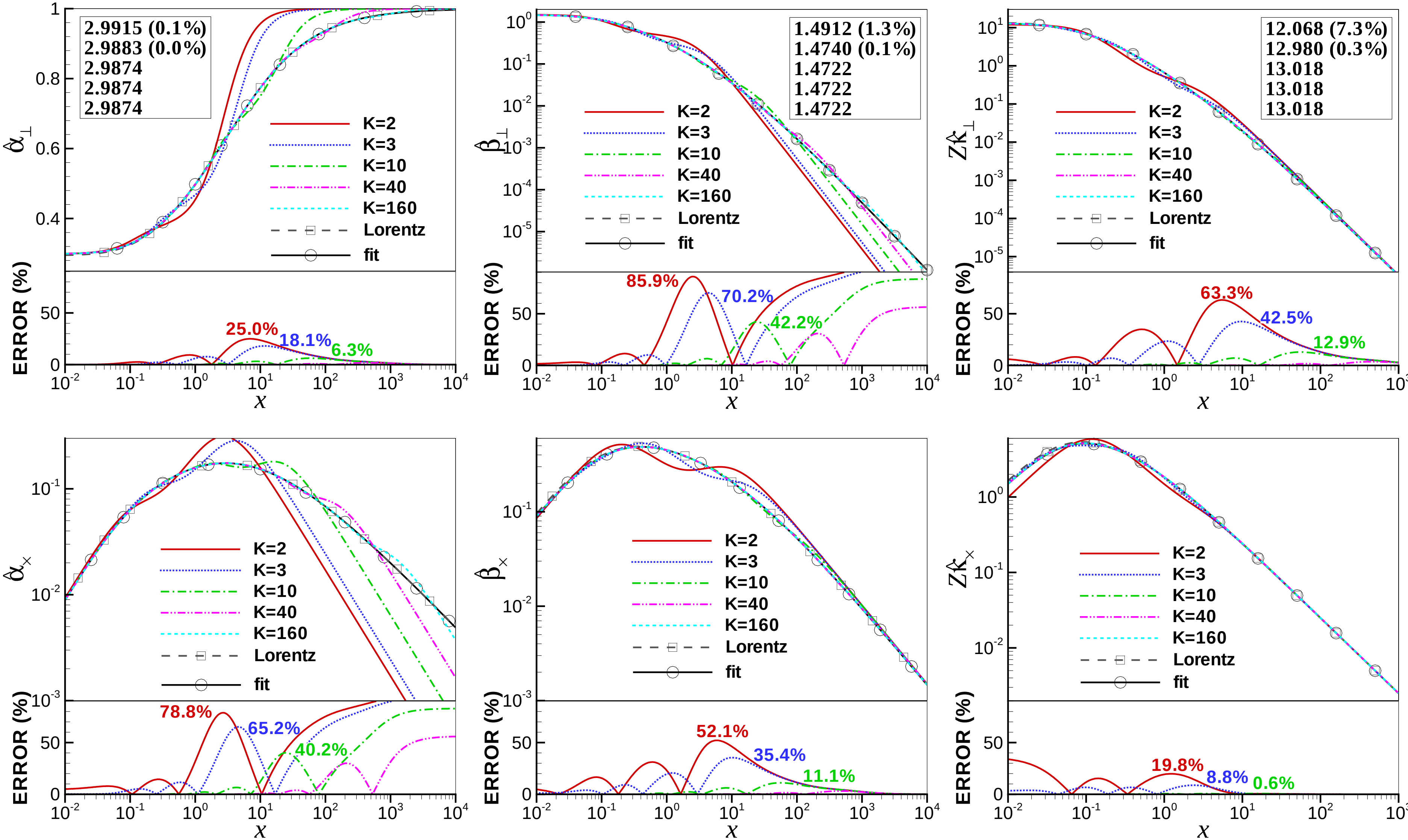
$$\mathbf{h}_\mathrm{e} = T_\mathrm{e}(\beta_\parallel \mathbf{V}_{\mathrm{ei}\parallel} + \hat{\beta}_\perp \mathbf{V}_{\mathrm{ei}\perp} + \hat{\beta}_\times \mathbf{V}_{\mathrm{ei}\times}) - \kappa_\parallel \nabla_\parallel T_\mathrm{e} - \kappa_\perp \nabla_\perp T_\mathrm{e} - \kappa_\times \nabla_\times T_\mathrm{e}$$



Convergence study with increasing moments $(Z = 100)$

$$\mathbf{R}_\mathrm{e} = -\alpha_\parallel \mathbf{V}_{\mathrm{ei}\parallel} - \alpha_\perp \mathbf{V}_{\mathrm{ei}\perp} + \alpha_\times \mathbf{V}_{\mathrm{ei}\times} - \beta_\parallel \nabla_\parallel T_\mathrm{e} - \beta_\perp \nabla_\perp T_\mathrm{e} - \beta_\times \nabla_\times T_\mathrm{e}$$

$$\mathbf{h}_\mathrm{e} = T_\mathrm{e}(\beta_\parallel \mathbf{V}_{\mathrm{ei}\parallel} + \hat{\beta}_\perp \mathbf{V}_{\mathrm{ei}\perp} + \hat{\beta}_\times \mathbf{V}_{\mathrm{ei}\times}) - \kappa_\parallel \nabla_\parallel T_\mathrm{e} - \kappa_\perp \nabla_\perp T_\mathrm{e} - \kappa_\times \nabla_\times T_\mathrm{e}$$



Electron coefficients (160 moment sol.+asymp. sol.)

	Braginskii	Epperlein & Haines (1986)	this work
$\hat{\alpha}_\perp$	$1-\dfrac{\alpha_1' x^2+\alpha_0'}{x^4+\delta_1 x^2+\delta_0}$	$1-\dfrac{\alpha_1' x+\alpha_0'}{x^2+a_1' x+a_0'}$	$1-\dfrac{1.462 Z^{\frac{2}{3}} r+\alpha_0(1-\hat{\alpha}_\parallel)}{r^{\frac{5}{3}}+\alpha_2 r^{\frac{4}{3}}+\alpha_1 r+\alpha_0}$
$\hat{\alpha}_\times$	$\dfrac{x(\alpha_1'' x^2+\alpha_0'')}{x^4+\delta_1 x^2+\delta_0}$	$\dfrac{x(\alpha_1'' x+\alpha_0'')}{(x^3+a_2'' x^2+a_1'' x+a_0'')^{8/9}}$	$\dfrac{Z^{\frac{2}{3}} r(2.532 r+a_0/a_5)}{r^{\frac{8}{3}}+a_4 r^{\frac{7}{3}}+a_3 r^2+a_2 r^{\frac{5}{3}}+a_1 r+a_0}$
$\hat{\beta}_\perp$	$\dfrac{\beta_1' x^2+\beta_0'}{x^4+\delta_1 x^2+\delta_0}$	$\dfrac{\beta_1' x+\beta_0'}{(x^3+b_2' x^2+b_1' x+b_0')^{8/9}}$	$\dfrac{6.330 Z^{\frac{5}{3}} r+\beta_0 \hat{\beta}_\parallel}{r^{\frac{8}{3}}+\beta_4 r^{\frac{7}{3}}+\beta_3 r^2+\beta_2 r^{\frac{5}{3}}+\beta_1 r+\beta_0}$
$\hat{\beta}_\times$	$\dfrac{x(\beta_1'' x^2+\beta_0'')}{x^4+\delta_1 x^2+\delta_0}$	$\dfrac{x(\beta_1'' x+\beta_0'')}{x^3+b_2'' x^2+b_1'' x+b_0''}$	$\dfrac{Zr(\frac{3}{2} r+b_0/b_5)}{r^3+b_4 r^{\frac{7}{3}}+b_3 r^2+b_2 r^{\frac{5}{3}}+b_1 r+b_0}$
$\hat{\kappa}_\perp$	$\dfrac{\gamma_1' x^2+\gamma_0'}{x^4+\delta_1 x^2+\delta_0}$	$\dfrac{\gamma_1' x+\gamma_0'}{x^3+c_2' x^2+c_1' x+c_0'}$	$\dfrac{(\frac{14}{3} Z+\sqrt{2}) r+\kappa_0 \hat{\kappa}_\parallel^{\mathrm{e}}}{r^3+\kappa_4 r^{\frac{7}{3}}+\kappa_3 r^2+\kappa_2 r^{\frac{5}{3}}+\kappa_1 r+\kappa_0}$
$\hat{\kappa}_\times$	$\dfrac{x(\gamma_1'' x^2+\gamma_0'')}{x^4+\delta_1 x^2+\delta_0}$	$\dfrac{x(\gamma_1'' x+\gamma_0'')}{x^3+c_2'' x^2+c_1'' x+c_0''}$	$\dfrac{r(\frac{5}{2} r+k_0/k_5)}{r^3+k_4 r^{\frac{7}{3}}+k_3 r^2+k_2 r^{\frac{5}{3}}+k_1 r+k_0}$
Error		less than 15 %	less than 1%
$Z=$	$1,2,3,4,\infty$	$1-8,10,12,14,20,30,60,\infty$	arbitrary (function of Z)

Coefficients of the rational polynomials of α_A, β_A, and κ_A

	$Z=1$	$Z=2$	arbitrary Z		$Z=1$	$Z=2$	arbitrary Z
$\hat{\alpha}_\parallel$	0.504	0.431	$1 - \frac{Z^{\frac{2}{3}}}{1.46Z^{\frac{2}{3}} - 0.330Z^{\frac{1}{3}} + 0.888}$	a_1	11.22	21.27	$2.18Z^{\frac{5}{3}} + 5.31Z + 3.73$
α_0	2.13	3.078	$0.623Z^{\frac{5}{3}} - 2.61Z^{\frac{4}{3}} + 3.56Z + 0.557$	a_2	7.35	15.41	$7.41Z + 1.11Z^{\frac{2}{3}} - 1.17$
α_1	2.97	3.997	$2.24Z^{\frac{2}{3}} - 1.11Z^{\frac{1}{3}} + 1.84$	a_3	6.14	7.253	$3.89Z^{\frac{2}{3}} - 4.51Z^{\frac{1}{3}} + 6.76$
α_2	-0.081	-0.106	$-0.0983Z^{\frac{1}{3}} + 0.0176$	a_4	2.541	3.128	$2.26Z^{\frac{1}{3}} + 0.281$
a_0	4.093	9.250	$0.0759Z^{\frac{8}{3}} + 0.897Z^2 + 2.06Z + 1.06$	a_5	5.07	9.671	$1.18Z^{\frac{5}{3}} - 1.03Z^{\frac{4}{3}} + 3.60Z + 1.32$
$\hat{\beta}_\parallel$	0.702	0.905	$\frac{Z^{\frac{5}{3}}}{0.693Z^{\frac{5}{3}} - 0.279Z^{\frac{4}{3}} + Z + 0.01}$	b_0	1.074	1.980	$\frac{6.87Z^3 + 78.2Z^2 + 623Z + 366}{1000}$
β_0	3.52	10.55	$0.156Z^{\frac{8}{3}} + 0.994Z^2 + 3.21Z - 0.84$	b_1	1.281	2.66	$0.134Z^2 + 0.977Z + 0.17$
β_1	8.23	20.03	$3.69Z^{\frac{5}{3}} + 3.77Z + 0.77$	b_2	4.896	6.746	$0.689Z^{\frac{4}{3}} - 0.377Z^{\frac{2}{3}} + 3.94Z^{\frac{1}{3}} + 0.644$
β_2	5.31	13.87	$9.43Z + 4.22Z^{\frac{2}{3}} - 12.9Z^{\frac{1}{3}} + 4.56$	b_3	-2.29	-2.605	$-0.109Z + 1.33Z^{\frac{2}{3}} - 3.80Z^{\frac{1}{3}} + 0.289$
β_3	3.99	5.955	$2.70Z^{\frac{2}{3}} + 1.46Z^{\frac{1}{3}} - 0.17$	b_4	2.98	4.425	$2.46Z^{\frac{2}{3}} + 0.522$
β_4	2.75	3.421	$2.58Z^{\frac{1}{3}} + 0.17$	b_5	1.131	2.202	$0.102Z^2 + 0.746Z + 0.072Z^{\frac{1}{3}} + 0.211$
$\hat{\kappa}^{\mathrm{e}}_\parallel$	3.204	2.464	$\frac{13.5Z^2 + 54.4Z + 25.2}{Z^3 + 8.35Z^2 + 15.2Z + 4.51}$	k_0	0.222	0.269	$\frac{0.0396Z^3 + 46.3Z + 176}{1000}$
κ_0	0.936	1.749	$\frac{9.91Z^3 + 75.3Z^2 + 518Z + 333}{1000}$	k_1	0.343	0.580	$\frac{15.4Z^3 + 188Z^2 + 240Z + 35.3}{1000Z + 397}$
κ_1	1.166	5.644	$\frac{0.211Z^3 + 12.7Z^2 + 48.4Z + 6.45}{Z + 57.1}$	k_2	0.655	0.252	$\frac{-0.159Z^2 - 12.5Z + 34.1}{Z^{\frac{2}{3}} + 0.741Z^{\frac{1}{3}} + 31.0}$
κ_2	-1.635	-2.212	$\frac{0.932Z^{\frac{7}{3}} + 0.135Z^2 + 12.3Z + 8.77}{Z + 4.84}$	k_3	0.899	1.626	$\frac{0.431Z^2 + 3.69Z + 0.0314}{Z + 3.62}$
κ_3	-1.635	-2.212	$\frac{0.246Z^3 + 2.65Z^2 - 92.8Z - 1.96}{Z^2 + 19.9Z + 35.3}$	k_4	-0.110	-0.201	$\frac{0.0258Z^2 - 1.63Z + 0.711}{Z^{\frac{4}{3}} + 4.36Z^{\frac{2}{3}} + 2.75}$
κ_4	2.370	4.129	$\frac{2.76Z^{\frac{5}{3}} - 0.836Z^{\frac{2}{3}} - 0.0611}{Z - 0.214}$	k_5	6.027	3.918	$\frac{Z^3 + 11.9Z^2 + 28.8Z + 9.07}{173Z + 133}$

Ordering for ion transport $\epsilon \sim \frac{v_T \tau_{ii}}{|\nabla|^{-1}} \ll 1$

- Transport ordering $\frac{\partial f}{\partial t} = O(\epsilon^2)? \Rightarrow \partial_t \mathsf{n}_i^{jp},\ \partial_t \mathbf{V}_i(\cdot)\mathsf{n}_i^{j\pm1,p},\ (\partial_t \ln T_i)\mathsf{n}_i^{jp}$

  Use $\frac{n_e}{\tau_{ei}} = \frac{1}{Z}\sqrt{\frac{1}{\mu\theta^3}}\frac{n_i}{\tau_{ii}}, \quad \mu \equiv \frac{m_e}{m_i}, \quad \theta \equiv \frac{T_e}{T_i}$ to write

$$\frac{3}{2}n_i d_i \ln T_i + \cdots = -3\frac{n_i}{\tau_{ii}}[\frac{1}{Z}\sqrt{\frac{\mu}{\theta}}(\frac{1}{\theta}-1)] \quad \left[= -3\frac{m_e}{m_i}\frac{n_e}{\tau_{ei}}(\frac{T_i}{T_e}-1)\right]$$

$$\underbrace{m_i n_i d_i \mathbf{V}_i - n_i q_i(\mathbf{E} + \mathbf{V}_i \times \mathbf{B})}_{-m_i n_i \mathbf{a}_i} + \cdots = \frac{m_e}{Z}\sqrt{\frac{1}{\mu\theta^3}}\frac{n_i}{\tau_{ii}}\mathbf{V}_{ei} + \cdots$$

$$\underbrace{d_i \mathsf{n}_i^{jp}}_{\epsilon^2} + \Omega_i \mathbf{b}\breve{\times}\mathsf{n}_i^{jp} + \{\hat{\Xi}^j \underbrace{(d_i \ln T)}_{\sqrt{\mu/\theta^3}} + \underbrace{\hat{U}_c^j \nabla\cdot\mathbf{V} + \hat{U}_l^j(\underline{\nabla}\mathbf{V})\cdot + \hat{U}_r^j(\nabla\underline{\mathbf{V}})\cdot}_{\epsilon}\}_{pk}\mathsf{n}_i^{jk}$$

$$+ \{\underbrace{v_T \hat{\Psi}^{j\pm}\nabla + v_T \hat{\Phi}^{j\pm}\nabla \ln T}_{\epsilon} + \hat{\Theta}^{j\pm}\underbrace{v_T^{-1}\mathbf{a}_i}_{\sqrt{\mu}\hat{V}}\}_{pk}(\cdot)\mathsf{n}_i^{j\pm1,k}$$

$$+ \hat{U}_{pk}^{j\pm}\underbrace{\nabla\mathbf{V}}_{\epsilon}(:)\mathsf{n}_i^{j\pm2,k} = \frac{1}{\tau_{ii}}(c_{ii}^{jpk} + Z a_{ie}^{jpk})\mathsf{n}_i^{jk} + G_i^{jp} + \underbrace{\hat{B}_{ie}^{jpk}}_{\mu}\mathsf{n}_e^{jk} + \underbrace{C_i^{(2)jp}}_{\epsilon^2}$$

- ▶ $\Omega_i \mathbf{b}\breve{\times}\mathsf{n}_i^{jp} + \hat{\Xi}_{pk}^j (d_i \ln T)\mathsf{n}_i^{jk} = (\hat{C}_{ii}^{jpk} + \hat{A}_{ie}^{jpk})\mathsf{n}_i^{jk} + G_i^{jp}$

Effective ion collision operator (for $T_\mathrm{i} \gtrsim T_\mathrm{e}$)

$$\Omega_\mathrm{i}\mathbf{b}\breve{\times}\mathsf{n}_\mathrm{i}^{jp} + \hat{\Xi}^j_{pk}(d_\mathrm{i}\ln T)\mathsf{n}_\mathrm{i}^{jk} = \frac{1}{\tau_\mathrm{ii}}(c_\mathrm{ii}^{jpk} + \frac{1}{Z}a_\mathrm{ie}^{jpk})\mathsf{n}_\mathrm{i}^{jk} + G_\mathrm{i}^{jp}$$

$$r\mathbf{b}\breve{\times}\mathsf{n}_\mathrm{i}^{jp}\mathsf{n}_\mathrm{i}^{jk} = [c_\mathrm{ii}^{jpk} + \underbrace{\frac{1}{Z}a_\mathrm{ie}^{jpk} - \hat{\Xi}^j_{pk}\tau_\mathrm{ii}(d_\mathrm{i}\ln T)}_{\frac{1}{Z}a_{\mathrm{i,eff}}}]\mathsf{n}_\mathrm{i}^{jk} + g_\mathrm{i}^{jp}$$

$$d_\mathrm{i}\ln T_\mathrm{i} + \cdots = -\frac{2}{\tau_\mathrm{ii}}[\frac{1}{Z}(\sqrt{\frac{\mu}{\theta^3}} - \sqrt{\frac{\mu}{\theta}})], \qquad \theta = \frac{T_\mathrm{e}}{T_\mathrm{i}}$$

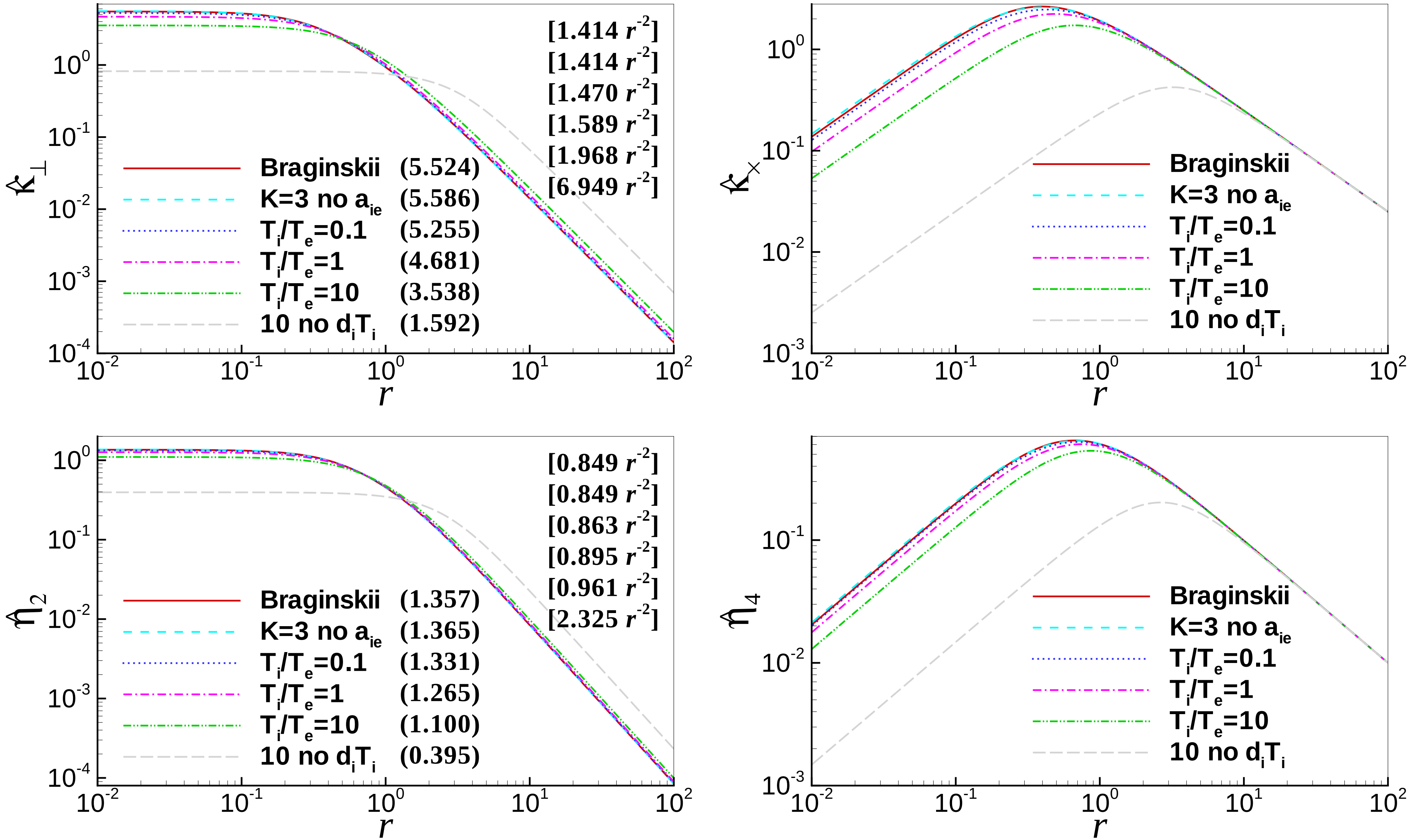
$$a^1_\mathrm{ie} = -\begin{pmatrix} 3\sqrt{\frac{\mu}{\theta^3}} & 0 & 0 \\ 2\sqrt{7}(\sqrt{\frac{\mu}{\theta}} - \sqrt{\frac{\mu}{\theta^3}}) & 5\sqrt{\frac{\mu}{\theta^3}} & 0 \\ 0 & 3\sqrt{6}(\sqrt{\frac{\mu}{\theta}} - \sqrt{\frac{\mu}{\theta^3}}) & 7\sqrt{\frac{\mu}{\theta^3}} \end{pmatrix}$$

$$2\hat{\Xi}^1 = \begin{pmatrix} 3 & 0 & 0 \\ -2\sqrt{7} & 5 & 0 \\ 0 & -3\sqrt{6} & 7 \end{pmatrix}, \quad a^1_{\mathrm{i,eff}} = -\begin{pmatrix} 3\sqrt{\frac{\mu}{\theta}} & 0 & 0 \\ 0 & 5\sqrt{\frac{\mu}{\theta}} & 0 \\ 0 & 0 & 7\sqrt{\frac{\mu}{\theta}} \end{pmatrix}$$

- Moment equations (3×3) for high collisionality

$$\begin{pmatrix} -r\mathbf{b}\times\mathsf{n}^{11} \\ -r\mathbf{b}\times\mathsf{n}^{12} \\ -r\mathbf{b}\times\mathsf{n}^{13} \end{pmatrix} = \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} \\ c^1_{21} & c^1_{22} & c^1_{23} \\ c^1_{31} & c^1_{32} & c^1_{33} \end{pmatrix}\begin{pmatrix} \mathsf{n}^{11} \\ \mathsf{n}^{12} \\ \mathsf{n}^{13} \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{5}}{2}\hat{\nabla}\ln T \\ 0 \\ 0 \end{pmatrix}$$

Ion transport coefficients from 3×3 calculation



General closures in small gyroradius ordering $\delta = \rho/L \ll 1$

$$\hat{D}\mathsf{n} + \underbrace{\Omega}_{\delta^{-1}} \mathbf{b}\breve{\times}\mathsf{n} = \hat{C}\mathsf{n} + \mathsf{G}\ (\mathsf{G} = \mathsf{G}^{(0)} + \mathsf{G}^{(1)}\ \text{FS})\ \text{with}\ \mathsf{n} = \mathsf{n}^{(0)} + \mathsf{n}^{(1)} + \mathsf{n}^{(2)} + \cdots$$

- δ^0 : $\Omega \mathbf{b}\breve{\times}\mathsf{n}^{(0)} = 0 \quad \Rightarrow \quad \mathsf{n}^{lk(0)} = \frac{(2l-1)!!}{l!} n_{\|}^{lk(0)} \mathsf{P}^l(\mathbf{b})$
- δ^1 : $\hat{D}\mathsf{n}^{(0)} + \Omega \mathbf{b}\breve{\times}\mathsf{n}^{(1)} = \hat{C}\mathsf{n}^{(0)} + \mathsf{G}^{(0)}$

  $$\bar{D}_{\|}\bar{n}_{\|}^{(0)} = \hat{C}\bar{n}_{\|}^{(0)} + \bar{G}_{\|}^{(0)} \Rightarrow n_{\|}^{jp(0)}(\ell) = \int d\ell' K^{jp,lk}(\ell-\ell') G^{lk(0)}(\ell')\ (\underbrace{=}_{\mathrm{FS}} 0)$$

  $$\Omega \mathbf{b}\breve{\times}\mathsf{n}^{l(1)} = C\mathsf{n}^{l(0)} + \mathsf{G}^{l(0)} - (\hat{D}\mathsf{n})^{l(0)} \Rightarrow \mathsf{n}_{\perp}^{l(1)}$$

- δ^2 : $\hat{D}\mathsf{n}^{(1)} + \Omega \mathbf{b}\breve{\times}\mathsf{n}^{(2)} = \hat{C}\mathsf{n}^{(1)} + \mathsf{G}^{(1)}$

  $$\bar{D}_{\|}\bar{n}_{\|}^{(1)} = \hat{C}\bar{n}_{\|}^{(1)} + \bar{G}_{\|,\mathrm{eff}}^{(1)}\ \text{where}\ \bar{G}_{\|,\mathrm{eff}}^{(1)} = \bar{G}_{\|}^{(1)} - (\hat{D}\mathsf{n}_{\perp}^{(1)})_{\|}$$

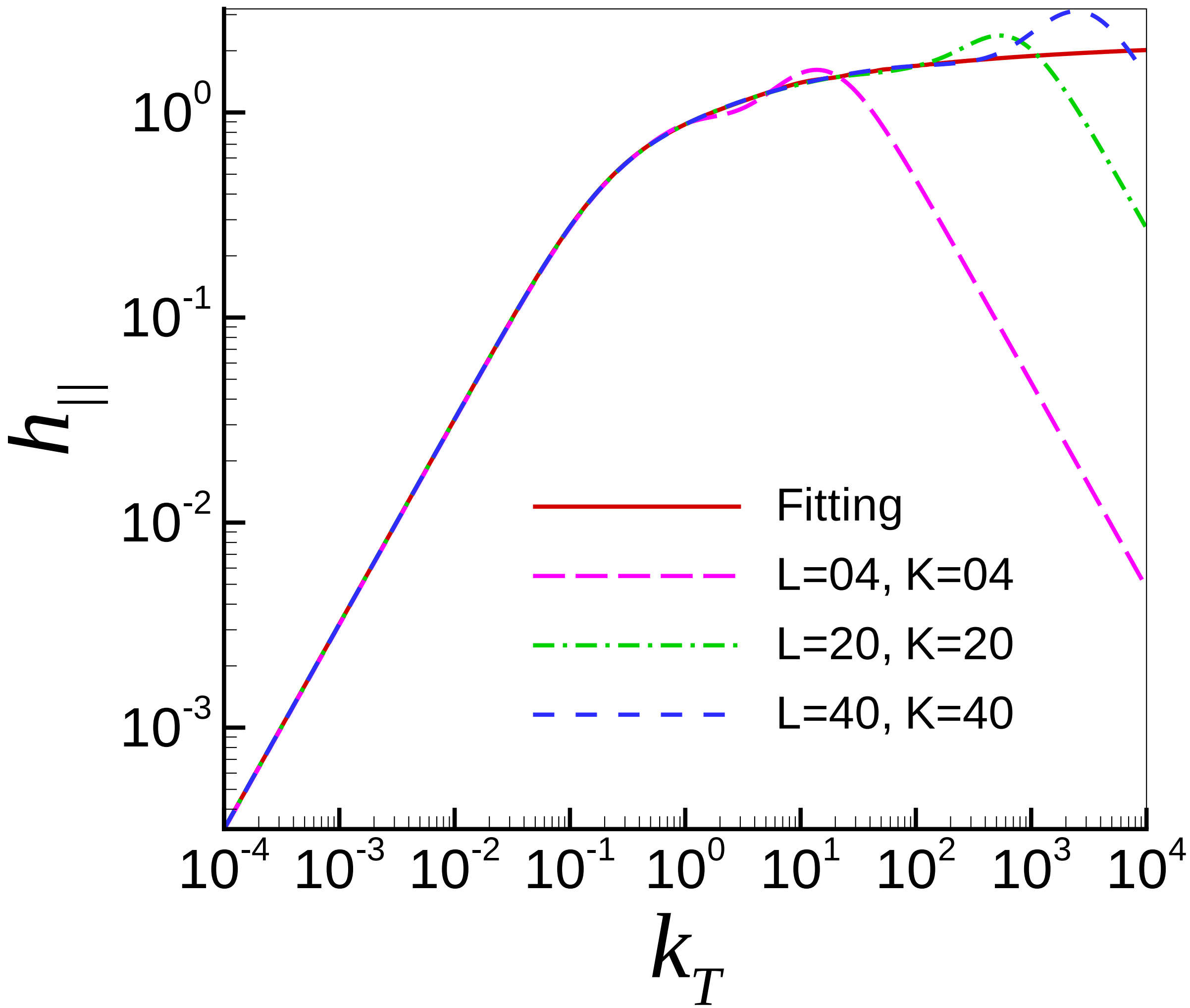
  $$\Rightarrow n_{\|}^{jp(1)}(\ell) = \int d\ell' K^{jp,lk}(\ell-\ell') G_{\mathrm{eff}}^{lk(1)}(\ell')$$

  $$(\hat{D}\mathsf{n})^{l(1)} + \mathbf{b}\breve{\times}\mathsf{n}^{l(2)} = C\mathsf{n}^{l(1)} + \mathsf{G}^{l(1)} \Rightarrow \mathsf{n}_{\perp}^{l(2)}$$

- ▶ No flux surface average, exact collision opeartor, general collisionality

Example: electron heat flow for $T(\ell) = T_0 + T_1 \sin k_T \ell$

A fitting formula for $K^{11,11}$ is constructed from the 1600 $(L = 40, K = 40)$ parallel moment solution and the collisionless limit



Obtain simple fitting formulas for other $K^{jp,lk}$

Summary (done) and future work

- Analytical solution for a uniform single component plasma
  - Ion-electron or ion-ion equilibration (with nonlinear collision terms)
  - General 21 moment equations in NIMROD
- Closures for high collisionality ion-electron plasmas (NIMROD)
  - General mass ratio (impurities)
- General collisionality: small gyroradius (δ) ordering
  - Parallel closures for general collisionality (heat flow: NIMROD SSPX)
    - Simple form of kernel functions
    - $\nabla_{||} \ln B$ effect
  - Up to δ^2 order in perpendicular transport on flux surfaces (no flux surface average)
  - Without flux surfaces
  - Apply to various fusion devices