Insulated Conductor Boundary Condition

At the surface of a conducting wall the tangential component of the electric field must vanish. The normal component of Faraday's law then requires that the normal component of the magnetic field cannot change and must vanish if it vanishes initially, which will be assumed.

$$\nabla_t \times \mathbf{E}_t = -\dot{\mathbf{B}}_n = 0$$

Let the conductor have a thin coating of insulator on its inside surface. Assume that the insulator is thin enough so that the normal component of the magnetic field will remain zero throughout. At the surface of the insulator Faraday's law requires the tangential electric field be the gradient of a surface potential.

$$\nabla_t \times \mathbf{E}_t = 0 \rightarrow \mathbf{E}_t = -\nabla_t \Phi$$

At the surface of the insulator there can be no normal current so Ampere's law requires that the tangential magnetic field be the gradient of a surface potential.

$$\nabla_t \times \mathbf{B}_t = \mathbf{J}_n = 0 \rightarrow \mathbf{B}_t = \nabla_t \Psi$$

If the plasma is resistive then the tangential component of Ohm's law at the surface is:

$$\mathbf{E}_t = \eta \mathbf{J}_t - v_n \hat{\mathbf{n}} \times \mathbf{B}_t$$

The normal component of Ohm's law at the surface is:

$$\mathbf{E}_n + \mathbf{v}_t \times \mathbf{B}_t = \eta \mathbf{J}_n = 0$$

The normal component of **E** must vanish if the tangential velocity vanishes, which will be assumed.

Assume that the currents at the vertices are computed from the magnetic field using a finite volume differencing formula:

$$\mathbf{J} = \frac{1}{V_v} \sum_{faces} \mathbf{S}_f \times \mathbf{B}_f$$

For vertices on the boundary, one of the faces of the vertex volume will be the boundary face and that term will be the boundary area vector crossed with the magnetic field at the boundary:

$$\mathbf{J} = \left(\frac{1}{V_v} \sum_{\substack{\text{interior} \\ faces}} \mathbf{S}_f \times \mathbf{B}_f \right) + \frac{\mathbf{S}_b \times \mathbf{B}_b}{V_v} \equiv \mathbf{J}^{\text{int}} + \frac{S_v}{V_v} \hat{\mathbf{n}} \times \mathbf{B}_t$$

Using this, the tangential components of Ohm's law at the surface become:

$$\mathbf{E}_t = \eta \mathbf{J}_t^{\text{int}} + \left(\eta S_v / V_v - v_n \right) \hat{\mathbf{n}} \times \mathbf{B}_t \qquad (1)$$

Using the potentials for the electric and magnetic fields, the tangential components of Ohm's law at the surface become:

$$-\nabla_t \Phi = \eta \mathbf{J}_t^{\mathrm{int}} + \left(\eta S_v / V_v - v_n \right) \hat{\mathbf{n}} \times \nabla_t \Psi$$

From this, it can be shown that the surface electric and magnetic potentials satisfy the Poisson-like equations:

$$\nabla_t \cdot \left(\frac{1}{\eta S_v / V_v - v_n} \nabla_t \Phi \right) = \nabla_t \cdot \left(\frac{\eta \mathbf{J}_t^{\mathrm{int}}}{\eta S_v / V_v - v_n} \right)$$

$$\nabla_t \cdot \left(\left(\eta S_v / V_v - v_n \right) \nabla_t \Psi \right) = -\nabla_t \cdot \left(\hat{\mathbf{n}} \times \eta \mathbf{J}_t^{\mathrm{int}} \right)$$

Circuit loop voltages or currents can be incorporated by adding jump conditions to the electric or magnetic potential equation.

Ignore the velocity and focus on the implicit diffusion equations for $\mathbf{A}$ and $\boldsymbol{\Psi}$:

$$\begin{pmatrix} (\sigma V_v/\Delta t)\,\mathbf{I} + V_v \mathbf{J}^{\text{int}} & \mathbf{S}_v \times \nabla_t \\ -\nabla_t \cdot \left(\hat{\mathbf{n}} \times \eta \mathbf{J}_t^{\text{int}}\right) & -\nabla_t \cdot \left(\eta S_v / V_v\right) \nabla_t \end{pmatrix} \begin{pmatrix} \mathbf{A} \\ \boldsymbol{\Psi} \end{pmatrix} = \begin{pmatrix} (\sigma V_v/\Delta t)\,\mathbf{A}_0 \\ 0 \end{pmatrix}$$

Diagonal terms are SPD but off-diagonals are unrelated and lower one has third derivatives, making this system of equations numerically unsolvable. Change to:

$$\begin{pmatrix} (\sigma V_v/\Delta t)\,\mathbf{I} + V_v \mathbf{J}^{\text{int}} & \mathbf{S}_v \times \nabla_t \\ -\mathbf{S}_v \cdot \nabla_t \times & 0 \end{pmatrix} \begin{pmatrix} \mathbf{A} \\ \boldsymbol{\Psi} \end{pmatrix} = \begin{pmatrix} (\sigma V_v/\Delta t)\,\mathbf{A}_0 \\ 0 \end{pmatrix}$$

This system of equations is fully SPD. (Not obvious.)

Proof of symmetry – start with integral identity:

$$\int d\mathbf{S}\cdot\nabla\times\left(\Psi\mathbf{A}\right)=\int\Psi d\mathbf{S}\cdot\nabla\times\mathbf{A}+\int d\mathbf{S}\cdot\nabla\Psi\times\mathbf{A}$$

$$\int\Psi\mathbf{A}\cdot d\mathbf{l}=\int\Psi\left(d\mathbf{S}\cdot\nabla\times\right)\mathbf{A}+\int\mathbf{A}\cdot\left(d\mathbf{S}\times\nabla\right)\Psi$$

The left hand side is zero if there are no gaps in the surface. If there are gaps then it reduces to a sum of the products of the currents and fluxes entering each gap. This allows coupling to external circuits.

Eigenvalues were computed in a torus using FEMLAB and found to be all positive for this system:

$$\begin{pmatrix}\left(\sigma V_v/\Delta t\right)\mathbf{I}+V_v\mathbf{J}^{\mathrm{int}} & \mathbf{S}_v\times\nabla_t\\ -\mathbf{S}_v\cdot\nabla_t\times & 0\end{pmatrix}\begin{pmatrix}\mathbf{A}\\ \Psi\end{pmatrix}=\lambda\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{A}\\ \Psi\end{pmatrix}$$

Lagrangian formalism: (circuit terms omitted)

$$L = \int \left(\frac{\Delta t \sigma}{2} \left(\frac{\mathbf{A} - \mathbf{A}_0}{\Delta t} \right)^2 + \frac{1}{2} \left(\nabla \times \mathbf{A} \right)^2 \right) dV + \int \Psi \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

$$\delta L = \int \left(\frac{\sigma}{\Delta t} \left(\mathbf{A} - \mathbf{A}_0 \right) \cdot \delta \mathbf{A} + \nabla \times \mathbf{A} \cdot \nabla \times \delta \mathbf{A} \right) dV$$
$$+ \int \left(\delta \Psi \nabla \times \mathbf{A} + \Psi \nabla \times \delta \mathbf{A} \right) \cdot d\mathbf{S}$$

$$\delta L = \int \delta \mathbf{A} \cdot \left(\frac{\sigma}{\Delta t} \left(\mathbf{A} - \mathbf{A}_0 \right) + \nabla \times \nabla \times \mathbf{A} \right) dV$$
$$+ \int \delta \mathbf{A} \cdot \left(\nabla \times \mathbf{A} - \nabla \Psi \right)_t \times d\mathbf{S} + \int \delta \Psi \left(\nabla \times \mathbf{A} \right)_n \cdot d\mathbf{S}$$

Analytically, all variations are arbitrary and independent, so all terms in brackets must separately vanish, giving 3 equations. Numerically, variations in **A** on boundary and in volume are related, so there are only 2 equations.