

# Impact of Velocity Space Distribution on Hybrid Kinetic-MHD Simulation of the (1, 1) Internal Kink Mode

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## NIMROD<sup>a</sup> (NonIdeal MHD with Rotation - Open Discussion)

- massively parallel 3-D MHD simulation
- finite elements in poloidal plane and Fourier modes in toroidal direction → axisymmetric geometry
- utilizes Lagrange type quadrilateral structured finite elements in 2-D
- can handle extreme anisotropies,  $\frac{\chi_{\parallel}}{\chi_{\perp}} \gg 1$
- flexibility to model general geometry → real experiments
- model experiment relevant parameters,  $S > 10^7$
- semi-implicit advance, not restricted by magnetosonic CFL condition
- assumes a steady state background and evolves perturbed quantities  
→  $A(\mathbf{x}, t) = A_s(\mathbf{x}) + \delta A(\mathbf{x}, t)$
- allows linear and nonlinear simulations



## NIMROD equations

- NIMROD evolves the **extended** MHD equations

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\ \mathbf{E} &= -\mathbf{U} \times \mathbf{B} + \eta \mathbf{J} + \frac{1}{ne} \mathbf{J} \times \mathbf{B} \\ &\quad + \frac{m_e}{ne^2} \left[ \frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot (\mathbf{J}\mathbf{U} + \mathbf{U}\mathbf{J}) \right. \\ &\quad \left. + \sum_{\alpha} \frac{q_{\alpha}}{m_{\alpha}} (\nabla p_{\alpha} + \nabla \cdot \Pi_{\alpha}) \right]\end{aligned}$$

$$\begin{aligned}\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{U}) &= \nabla \cdot D\nabla n \\ mn \left( \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) &= \mathbf{J} \times \mathbf{B} - \nabla p - \nabla \cdot \Pi - \nabla \cdot p_h\end{aligned}$$

$$\begin{aligned}\frac{n_{\alpha}}{\gamma - 1} \left( \frac{\partial T_{\alpha}}{\partial t} + \mathbf{U}_{\alpha} \cdot \nabla T_{\alpha} \right) &= -\nabla \cdot q_{\alpha} + Q_{\alpha} \\ &\quad - p_{\alpha} \nabla \cdot \mathbf{U}_{\alpha} - \Pi_{\alpha} : \nabla \mathbf{U}_{\alpha}\end{aligned}$$

## Numeric representation of NIMROD fields

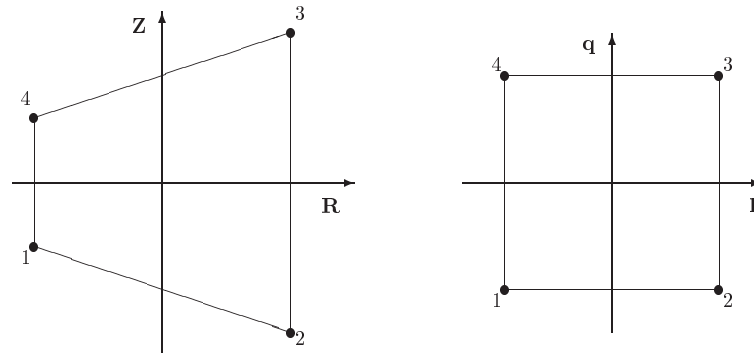
- the **perturbed** NIMROD fields are in FE-Fourier representation

$$\delta A(\mathbf{x}, t) = \sum_j A_{j,0}(t) \alpha_{j,0} + \sum_j \sum_n (A_{j,n}(t) \alpha_{j,n} + c.c.)$$

where

$$\alpha_{j,n} = N_j(p, q) \exp(in\phi)$$

$(p, q)$  are logical coordinates



- structured finite elements
- PIC becomes nontrivial in FE

## PIC in FEM

- gather/scatter require logical coordinates
- FE representation for  $(R, Z)$  need to be inverted

$$R = \sum_j R_j N_j(p, q), \quad Z = \sum_j Z_j N_j(p, q),$$

- Use Newton method to solve for  $(p, q)$  given  $(R, Z)$

$$\begin{pmatrix} p^{k+1} \\ q^{k+1} \end{pmatrix} = \begin{pmatrix} p^k \\ q^k \end{pmatrix} + A(p_k, q_k) \begin{pmatrix} R - R^k \\ Z - Z^k \end{pmatrix}$$

- Jacobian gives gradient of shape functions for free
- sorting is also required for parallelization
- efficient load balancing is tricky

## Algorithmic Details

- **nonuniform loading** in physical and velocity space
- want to match numeric load with analytic distribution for  $\delta f$
- particles loaded to match slowing down distribution
  - physical space loaded proportional to fluid pressure profile
  - velocity space loads slowing down distribution by inverting distribution

$$D(\varepsilon) = \frac{\int_0^\varepsilon f(\varepsilon') d\varepsilon'}{\int_0^{\varepsilon_{max}} f(\varepsilon') d\varepsilon'}$$

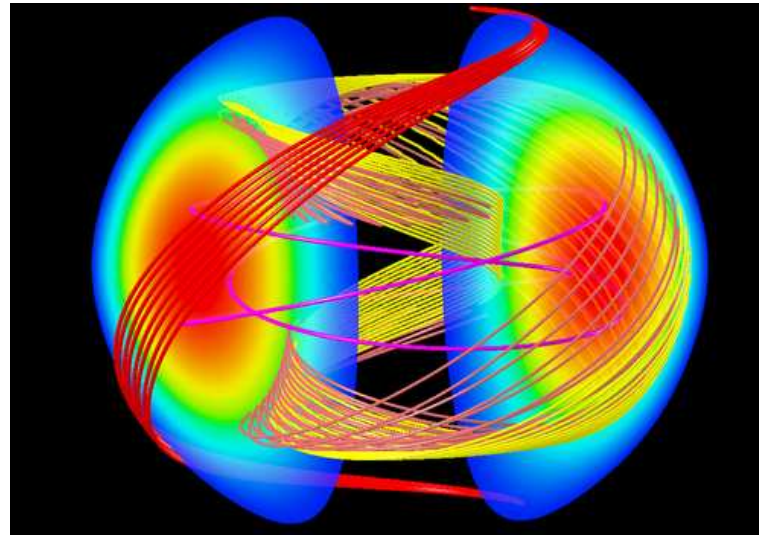
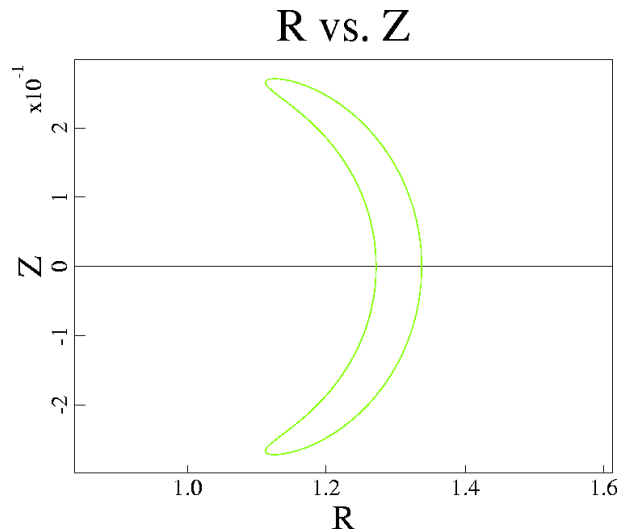
$D(\varepsilon)$  is a uniform random number where  $D(0) = 0$ ,  $D(\varepsilon_{max}) = 1$

- no equilibrium hot particle pressure anisotropy
- implement **integration by parts** of weak form of gradient of hot particle pressure tensor

$$- \int d\mathbf{x} \mathbf{A}^* \cdot \nabla \cdot \underline{\mathbf{p}}_h = \int d\mathbf{x} (\nabla \mathbf{A}^*)^T : \underline{\mathbf{p}}_h$$

# Hybrid Kinetic-MHD - Building Bridges

- capture kinetic effects lost in MHD approximation
- lost kinetic effects can **significantly** effect MHD instabilities<sup>a</sup>
  - fraction of plasma executes non-trivial orbits - large gyro-orbits, banana-orbits, potatoes



- usually high energy tail
- wave-particle interactions

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<sup>a</sup>W. Park, et al, "Three-dimensional hybrid gyrokinetic- magnetohydrodynamic simulation", *Physics of Fluids B*, 4, 1992

## The Hybrid Kinetic-MHD Equations<sup>a</sup>

- in the limit  $n_h \ll n_0$ ,  $\beta_h \sim \beta_0$ , quasi neutrality, only modification of MHD equations is addition of the **hot particle pressure tensor** in the momentum equation:

$$\rho \left( \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) = \mathbf{J} \times \mathbf{B} - \nabla p_b - \nabla \cdot \underline{\mathbf{p}}_h$$

the subscripts  $b, h$  denote the bulk plasma and hot particles

- the steady state equation

$$\mathbf{J}_0 \times \mathbf{B}_0 = \nabla p_0 = \nabla p_{b0} + \nabla p_{h0}$$

- evolved momentum equation is ( $\mathbf{U}_s = 0$ )

$$\rho_s \frac{\partial \delta \mathbf{U}}{\partial t} = \mathbf{J}_s \times \delta \mathbf{B} + \delta \mathbf{J} \times \mathbf{B}_s - \nabla \cdot \delta \underline{\mathbf{p}}_b - \nabla \cdot \delta \underline{\mathbf{p}}_h$$

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<sup>a</sup>C.Z.Cheng, 'A Kinetic MHD Model for Low Frequency Phenomena', *J. Geophys. Res* **96**, 1991

## Deposition of $\delta \underline{\mathbf{p}}_h$ onto Finite Element grid

- assume CGL-like form  $\delta \underline{\mathbf{p}}_h = \begin{pmatrix} p_{\perp} & 0 & 0 \\ 0 & p_{\perp} & 0 \\ 0 & 0 & p_{\parallel} \end{pmatrix}$
- evaluate pressure moment at a position  $\mathbf{x}$  is

$$\begin{aligned} \delta p(\mathbf{x}) &= \int m(v - V_h)^2 \delta f(\mathbf{x}, \mathbf{v}) d^3v \\ &= \sum_{i=1}^N m(v_i - V_h)^2 g_0 w_i \delta^3(x - x_i) \end{aligned}$$

where sum is over the particles,  $m$  mass of the particle,  $g_0 w_i$  is the perturbed phase density, and  $V_h$  is the hot flow velocity

- discretize to the poloidal plane by performing a Fourier transform,<sup>a</sup>

$$\delta p_n(R, Z) = \sum_{i=1}^N m(v_i - V_h)^2 g_0 w_i \delta(R - R_i) \delta(Z - Z_i) e^{-in\phi_i}$$

<sup>a</sup>for large  $n$  runs, a conventional  $\phi$  deposition with a FFT is performed

- express lhs  $\delta p_n(R, Z)$  in the finite element basis,

$$\delta p_n(R, Z) = \sum_k \delta p_n^k N^k(\eta, \xi)$$

where sum  $k$  is over the basis functions and  $(\eta, \xi)$  are functions of  $(R, Z)$

- project onto the finite element space by casting in **weak form**:

$$\int N^l \sum_k \delta p_n^k N^k d^3x = \int N^l \sum_{i=1}^N m(v_i - V_h)^2 g_0 w_i \delta(R - R_i) \delta(Z - Z_i) e^{-in\phi_i} d^3x$$

$$\underline{\mathbf{M}} \delta p_n^k = \sum_{l \in k} \sum_{i=1}^N m(v_i - V_h)^2 g_0 w_i e^{-in\phi_i} N^l(\eta_i, \xi_i)$$

$\mathbf{M}$  is the finite element mass matrix

- **gather** is done using the same shape functions

$$A(\mathbf{x}_i) = \sum_l A^l N^l(\eta_i, \xi_i)$$

for some field quantity  $A$

## The $\delta f$ PIC method<sup>a b</sup>

- PIC is a Lagrangian simulation of phase space  $f(\mathbf{x}, \mathbf{v})$
- PIC evolves the  $f(\mathbf{x}(\mathbf{t}), \mathbf{v}(\mathbf{t}))$
- spatial grid is not inherently necessary, but very convenient!
- in principle,  $f(\mathbf{x}(\mathbf{t}), \mathbf{v}(\mathbf{t}))$  contains everything
- typically PIC is noisy, can't beat  $1/\sqrt{N}$
- $\delta f$  PIC **reduces the discrete particle noise** associated with conventional PIC
- Vlasov Equation

$$\frac{\partial f(\mathbf{z})}{\partial t} + \dot{\mathbf{z}} \cdot \frac{\partial f(\mathbf{z})}{\partial \mathbf{z}} = 0$$

$\mathbf{z}$  is the phase coordinate

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<sup>a</sup>S. E. Parker and W. W. Lee, 'A fully nonlinear characteristic method for gyro-kinetic simulation', *Physics of Fluids B*, **5**, 1993

<sup>b</sup>G. Hu and J. A. Krommes, "Generalized weighting scheme for  $\delta f$  particle simulation method", *Physics of Plasmas*, **1**, 1994

- split phase space distribution into steady state and evolving perturbation:

$$f = f_{eq}(\mathbf{z}) + \delta f(\mathbf{z}, t)$$

- $\delta f$  evolves along the characteristics  $\dot{\mathbf{z}}$  (control variates MC<sup>c</sup>)

$$\delta \dot{f} = -\tilde{\mathbf{z}} \cdot \frac{\partial f_{eq}}{\partial \mathbf{z}}$$

using  $\mathbf{z} = \mathbf{z}_{eq} + \tilde{\mathbf{z}}$  and  $\dot{\mathbf{z}}_{eq} \cdot \frac{\partial f_{eq}}{\partial \mathbf{z}} = 0$

- the drift kinetic equations of motion are used as the particle characteristics

$$\dot{\mathbf{x}} = v_{\parallel} \hat{\mathbf{b}} + \frac{m}{eB^4} \left( u^2 + \frac{v_{\perp}^2}{2} \right) \left( \mathbf{B} \times \nabla \frac{B^2}{2} \right) + \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\mu_0 m v_{\parallel}^2}{eB^2} \mathbf{J}_{\perp}$$

$$m \dot{v}_{\parallel} = -\hat{\mathbf{b}} \cdot (\mu \nabla B - e \mathbf{E})$$

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<sup>c</sup>A. Y. Aydemir, "A unified MC interpretation of particle simulations...", *Physics of Plasmas*, **1**, 1994

## Slowing Down Distribution for Hot Particles

- for the slowing down distribution function

$$f_{eq} = \frac{P_0 \exp\left(\frac{P_\zeta}{\psi_0}\right)}{\varepsilon^{3/2} + \varepsilon_0^{3/2}}$$

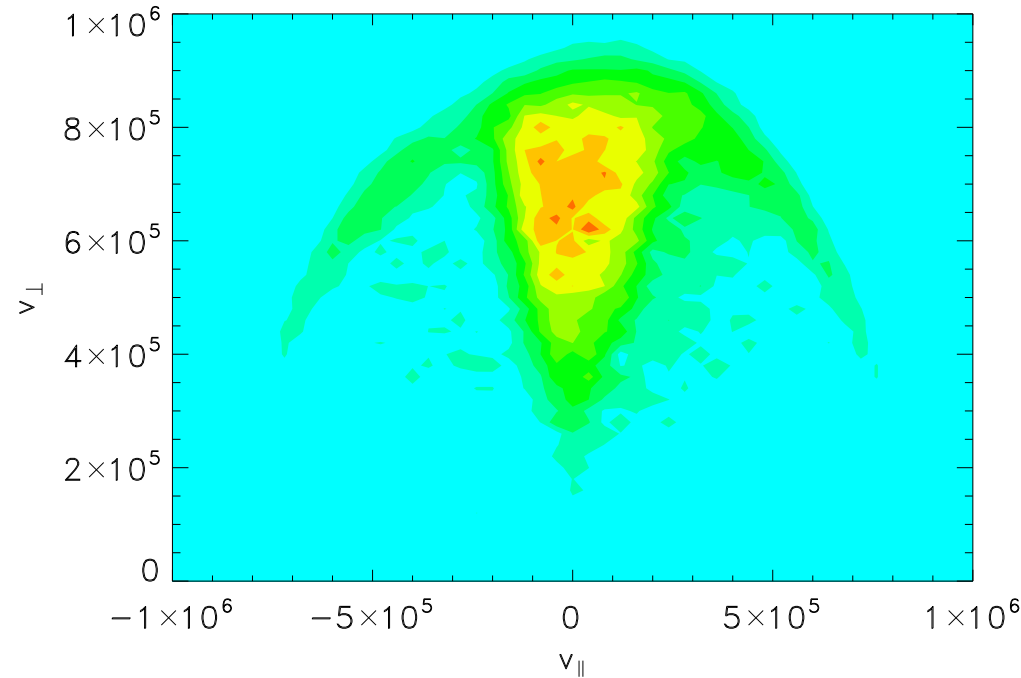
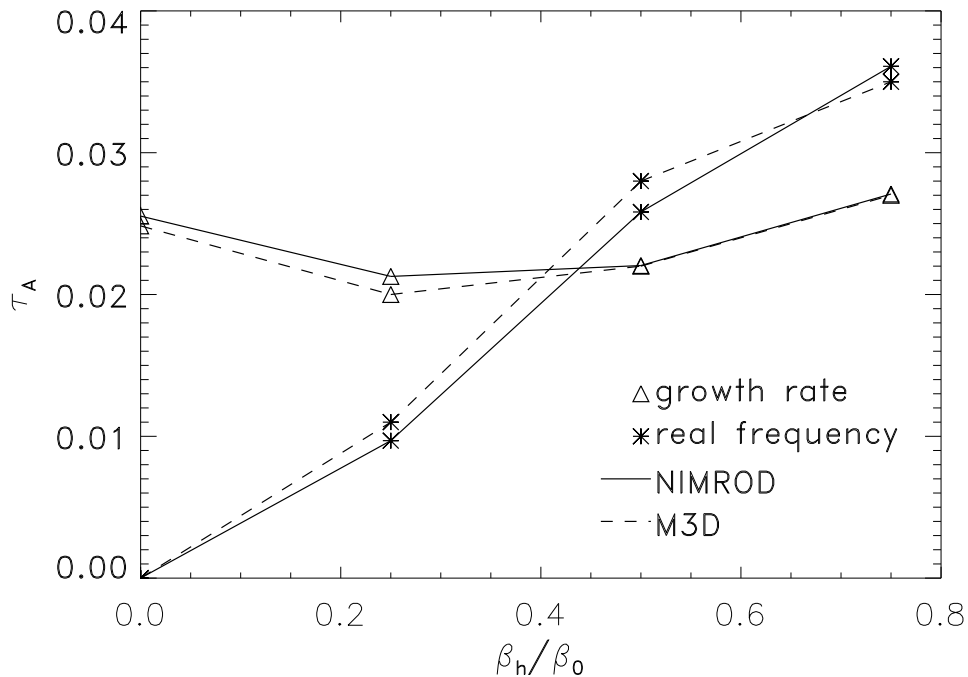
where  $P_\zeta = g\rho_{\parallel} - \psi$  is the canonical toroidal momentum and  $\varepsilon$  is the energy,  $\psi_0$  is the total flux, and  $\varepsilon_c$  is the critical slowing down energy

$$\begin{aligned} \delta \dot{f} = & f_{eq} \left\{ \frac{mg}{e\psi_0 B^3} \left[ \left( v_{\parallel}^2 + \frac{v_{\perp}^2}{2} \right) \delta \mathbf{B} \cdot \nabla B - \mu_0 v_{\parallel} \mathbf{J} \cdot \mathbf{E} \right] \right. \\ & \left. + \frac{\delta \mathbf{v} \cdot \nabla \psi_p}{\psi_0} + \frac{3}{2} \frac{e\varepsilon^{1/2}}{\varepsilon^{3/2} + \varepsilon_0^{3/2}} \mathbf{v}_D \cdot \mathbf{E} \right\} \end{aligned}$$

where

$$\begin{aligned} \mathbf{v}_D &= \frac{m}{eB^3} \left( v_{\parallel}^2 + \frac{v_{\perp}^2}{2} \right) (\mathbf{B} \times \nabla B) + \frac{\mu_0 m v_{\parallel}^2}{eB^2} \mathbf{J}_{\perp} \\ \delta \mathbf{v} &= \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \mathbf{v}_{\parallel} \cdot \frac{\delta \mathbf{B}}{B} \end{aligned}$$

## Benchmark with M3D



- improved particle loading - ensures velocity space isotropy
- velocity shows most activity in the most energetic particles
- mostly in trapped region, also in extremes of passing particles

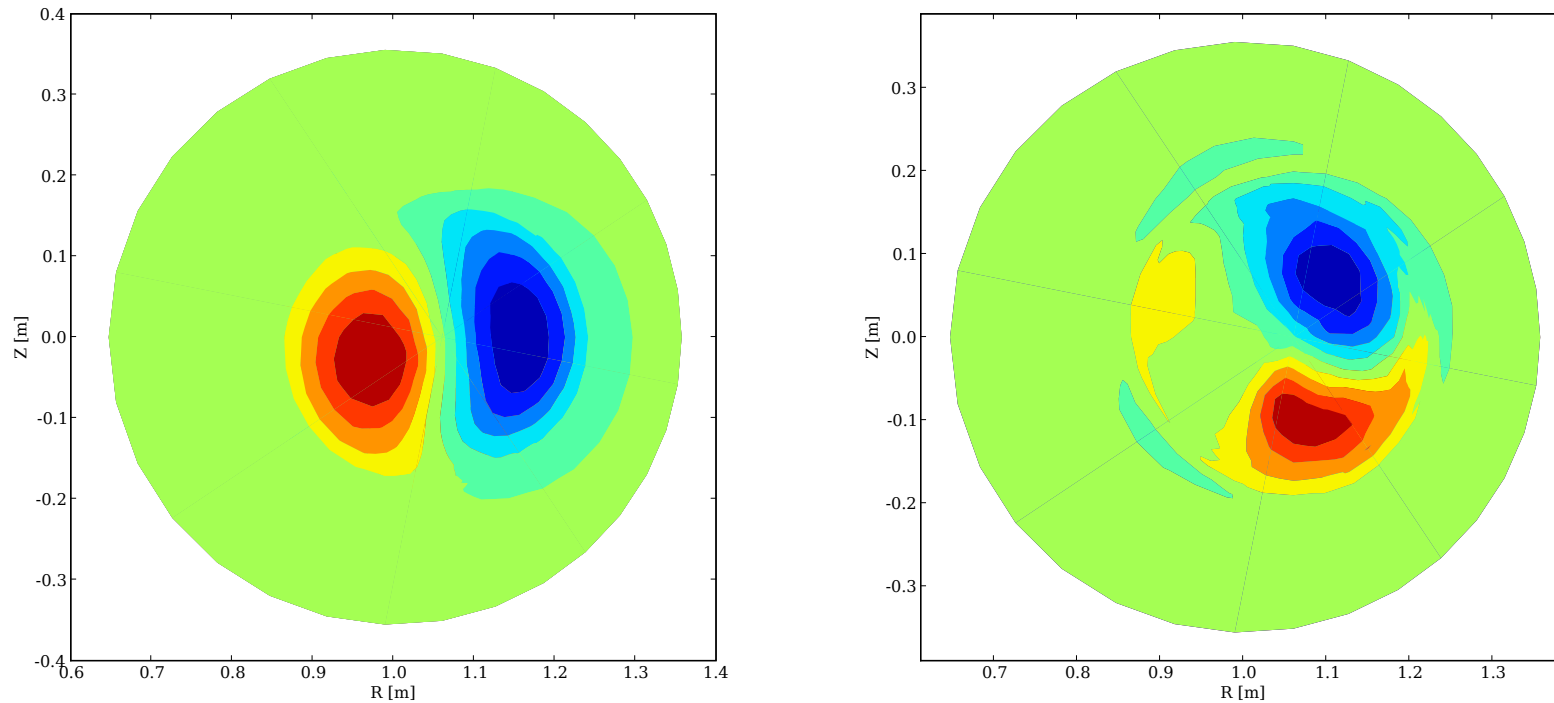


Figure 1:  $n = 1$   $p_{h\perp}$ ,  $\Delta p_h$

- simulations without anisotropy reproduce ideal MHD with  $\gamma$  within 10%
- no real frequency
- anisotropic pressure  $\Delta p = p_{\parallel} - p_{\perp}$  is key to energetic effects

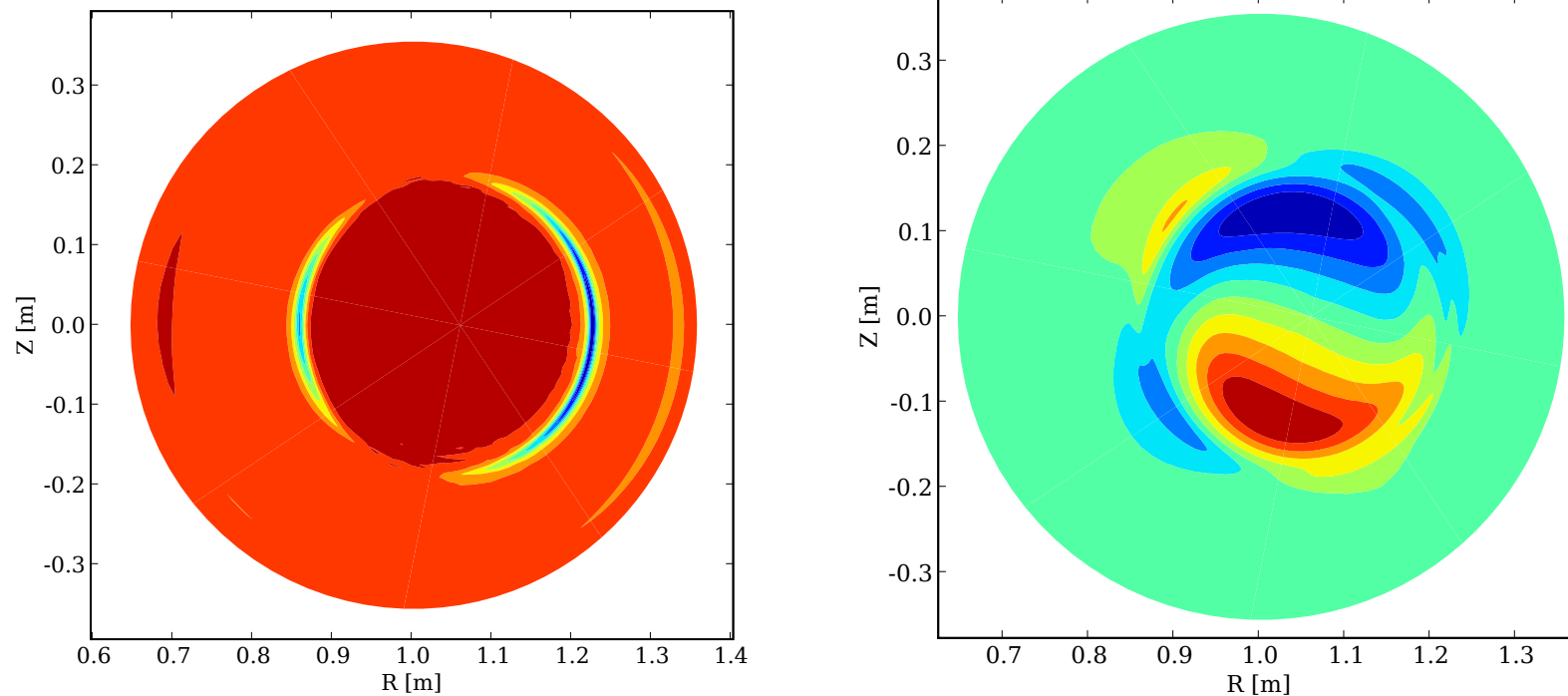


Figure 2:  $n = 1V_\phi$  with passing particles,  $V_\phi$  with trapped particles

- simulations with only passing particles stabilize but do not change  $V_\phi$  mode topology
- trapped particles excite precessional fishbone mode (plot on right)

## Impact of increasing velocity cutoff

- benchmark (1, 1) performed with  $v_{max} = 1 \times 10^6 m/s$
- increasing the velocity cutoff results in stronger stabilization
  - fixed  $\beta_h$  - increase in energy range, decrease in density → **fewer particles are doing more**
  - increase in real frequency

– examine  $\beta_h = .5$

$v_{max}$	$\gamma_{TA}$	$\omega_{TA}$
1.0	0.0218	0.0257
1.1	0.0180	0.0290
1.2	0.0142	0.0317
1.3	0.0093	0.0345
1.5	small	0.0370

## Examination of $\delta f$ in phase space

- more energetic particles drive higher frequency
- plots of velocity space density of  $n = 1$  projection of  $\delta f$  shows dominance of trapped particles as cut-off is increased
- this is reflected in decreasing relative amplitude of passing particle “wing”
- note the asymmetry in the profile
  - trapped cone region biased toward counter propagating (negative  $v_{||}$ )
  - passing region biased towards co-propagating

## Summary and Future Work

- modest increase in velocity cut-off has strong impact on stability
- velocity cut-off well below birth energy ( $100\text{KeV} \sim 3 \times 10^6$ )
- phase space plot shows dominance of trapped particles
- significance of phase space asymmetry needs to be explored
- impact of trapped vs. passing particles needs to be explored
- improve parallel efficiency
- improve numeric accuracy - higher order elements
- study impact on nonlinear evolution